

MAHLER MEASURE AND THE WZ ALGORITHM

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ABSTRACT. In this paper we will use the WZ algorithm to prove identities between Mahler measures of polynomials. In particular, we will offer a new proof of a theorem due to Lalín. We will also show that this theorem is equivalent to a formula for elliptic dilogarithms.

1. INTRODUCTION

In this paper, we will show that the Wilf-Zeilberger algorithm can be used to prove relations between Mahler measures of polynomials. The (logarithmic) Mahler measure of an n -variable Laurent polynomial, $P(x_1, \dots, x_n)$, is defined by

$$m(P) := \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n.$$

We will primarily be interested in values of the following special function:

$$m(\alpha) := m\left(\alpha + x + \frac{1}{x} + y + \frac{1}{y}\right).$$

In particular, there are many conjectured formulas linking special values of $m(\alpha)$ to the values of L -series of elliptic curves evaluated at $s = 2$. Deninger hypothesized that $m(1)$ should be a rational multiple of $L(E, 2)/\pi^2$, where E is a conductor 15 elliptic curve [9]. Boyd used numerical calculations to make the constant explicit:

$$m(1) \stackrel{?}{=} \frac{15}{4\pi^2} L(E, 2). \quad (1)$$

He also conjectured hundreds of other formulas for $m(\alpha)$ [8]. So far, only a small fraction of his formulas have been proved.

In general, it is often much easier to prove identities between Mahler measures, than to prove formulas relating them to L -functions. Although (1) is unproven, Lalín recently showed that

$$11m(1) = m(16), \quad (2)$$

using algebraic K -theory [15]. Rodriguez-Villegas first identified the connection with algebraic K -theory, by demonstrating that identities such as (2) often follow from finding relations in the K_2 groups of elliptic curves [18]. Zagier asked whether or not the relation $6m(1) = m(5)$ could be proved with elementary calculus [22, pg. 56]. By an elementary result of Kurokawa and Ochiai [14], $m(1) + m(16) = 2m(5)$, Zagier's problem amounts to finding an elementary proof of (2). In this paper, we

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will present precisely such a proof, answering Zagier's question. The most difficult part of the proof follows from (15), which we will derive using the Wilf-Zeilberger algorithm.

In the second portion of the paper, we will present several new q -series expansions for Mahler measures. For instance, if $\varphi(q)$ is Ramanujan's theta function, then

$$m\left(4\frac{\varphi^2(q)}{\varphi^2(-q)}\right) = \frac{4}{\pi} \sum_{n=-\infty}^{\infty} D(iq^n), \quad (3)$$

where $D(z)$ is the Bloch-Wigner dilogarithm. These sorts of formulas provide an easy way to translate Mahler measures into elliptic dilogarithms, and vice-versa. As a corollary, we will translate an exotic relation due to Bertin into an identity between hypergeometric functions [5] (see formulas (35) and (36)). We are hopeful that this line of research will eventually lead to WZ-proofs of exotic relations. Furthermore, as remarked upon in the conclusion, it seem likely that some of the mysterious (computationally discovered) formulas for $1/\pi^2$, may eventually be linked to the theory of higher regulators.

2. AN APPLICATION OF THE WZ METHOD

We will begin with a brief review of the WZ method. We will say that $F(n, k)$ is hypergeometric, if $F(n+1, k)/F(n, k)$ and $F(n, k+1)/F(n, k)$ are rational functions of n and k . Two hypergeometric functions are called a WZ-pair if they satisfy the following functional equation:

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k). \quad (4)$$

Wilf and Zeilberger proved that if $F(n, k)$ satisfies (4), then it is always possible to determine $G(n, k)$ (see [17] and [26]). Their algorithm has been implemented in Maple and Mathematica, and as a result it is possible to find WZ-pairs by systematically guessing values of $F(n, k)$.

Let us consider WZ-pairs where F and G are meromorphic functions of n and k . If we sum both sides of (4) from $n = 0$ to $n = \infty$, the left-hand side of the equation telescopes, and we have

$$-F(0, k) + \lim_{n \rightarrow \infty} F(n, k) = \sum_{n=0}^{\infty} G(n, k+1) - \sum_{n=0}^{\infty} G(n, k).$$

In instances where $F(0, k) = 0$, and $\lim_{n \rightarrow \infty} F(n, k) = 0$, this becomes

$$\sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} G(n, k+1).$$

It follows immediately that the series is periodic with respect to k . If the series also converges uniformly, and j is an integer, then we can write

$$\sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} \lim_{j \rightarrow \infty} G(n, k+j).$$

If $\lim_{j \rightarrow \infty} G(n, k+j)$ is independent of k , then we can conclude that for unrestricted k :

$$\sum_{n=0}^{\infty} G(n, k) = \text{constant}. \quad (5)$$

We will use this method to prove Theorem 2, which will imply Mahler measure formulas such as (2).

In order to apply the WZ-method to formulas such as (2), we need to relate Mahler measures to hypergeometric functions. We will use several of the identities summarized in [20]. If $r \in (0, 1]$, results from [14] and [19] show that:

$$m\left(\frac{4}{r}\right) = \log\left(\frac{4}{r}\right) - \sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{(r/4)^{2n}}{2n}, \quad (6)$$

$$m(4r) = 4 \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{(r/4)^{2n+1}}{2n+1}. \quad (7)$$

Notice that both of these sums depend upon the same binomial coefficients. Therefore, if we define s by

$$s := \frac{m(4/r)}{m(4r)},$$

we can form a linear combination of (6) and (7) to obtain

$$\log\left(\frac{4}{r}\right) = rs + \sum_{n=1}^{\infty} \frac{(2(1+rs)n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \left(\frac{r}{4}\right)^{2n}. \quad (8)$$

It follows that $(r, s) \in \bar{\mathbb{Q}}^2$ and $r \in (0, 1]$, if and only if (8) also gives an explicit formula for an algebraic hypergeometric series. By linearity, explicit cases of (8) immediately imply formulas for s . Notice that (8) diverges when $|r| > 1$.

Theorem 1. *The following formulas are true:*

$$2 \log(2) = 1 + \sum_{n=1}^{\infty} \frac{(4n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{4n}}, \quad (9)$$

$$3 \log(2) = 2 + \sum_{n=1}^{\infty} \frac{(6n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{6n}}, \quad (10)$$

$$8 \log(2) = \frac{11}{2} + \sum_{n=1}^{\infty} \frac{(15n+2)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{8n}}. \quad (11)$$

Furthermore, (10) is equivalent to

$$4m(2) = m(8), \quad (12)$$

and (11) is equivalent to

$$11m(1) = m(16). \quad (13)$$

Somewhat surprisingly, we have not been able to prove (12) with WZ techniques. While it seems likely that such a proof exists, the identity has so far proven intractable. This is surprising, since the K -theoretic proof of (12) is much easier than the K -theoretic proof of (13). In order to prove (13), we will first prove (15) with WZ techniques, and then use that formula to derive (11). It is interesting to note, that Mathematica can recognize (9), but not (10) or (11). This probably occurs because it is possible to derive (9) using Dougall's theorem [24]. We will also prove (9) with the WZ method.

Theorem 2. *The following identities are true:*

$$\pi \frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2\left(x+\frac{1}{2}\right)} = \sum_{n=0}^{\infty} \frac{(4n+2x+1)}{(2n+1)(n+x)} \frac{\left(\frac{1}{2}+x\right)_n}{(1+x)_n} \binom{2n}{n} \frac{1}{2^{2n}}, \quad (14)$$

$$4\pi \frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2\left(x+\frac{1}{2}\right)} = \sum_{n=0}^{\infty} \frac{(2(2n+1)^2(15n+2) + xP(n, x))}{(2n+1)(2n+x)(2n+x+1)^2} \frac{\left(\frac{1}{2}+x\right)_n^2}{\left(1+\frac{x}{2}\right)_n \left(\frac{1+x}{2}\right)_n} \binom{2n}{n} \frac{1}{2^{6n}}, \quad (15)$$

where $P(n, x) = (2n+1)(86n+19) + 4x(20n+7) + 12x^2$.

Proof. Although this proof is short, it should be mentioned that a substantial amount of work was required to find the necessary WZ-pairs. We will begin by proving (14). Let us define the Pochhammer symbol using $(x)_m := \Gamma(x+m)/\Gamma(x)$. Now consider the following WZ-pair:

$$\begin{aligned} F(n, k) &= -\frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_k^2}{(1+k)_n (1)_n (1)_k^2} \cdot \frac{n}{2(n+k)}, \\ G(n, k) &= \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_k^2}{(1+k)_n (1)_n (1)_k^2} \cdot \frac{k(4n+2k+1)}{2(n+k)(2n+1)}. \end{aligned} \quad (16)$$

It is easy to see that $F(0, k) = 0$, and $\lim_{n \rightarrow \infty} F(n, k) = 0$. Furthermore, $\sum_{n=0}^{\infty} G(n, k)$ converges uniformly, so by the previous discussion we may conclude that

$$\begin{aligned} \sum_{n=0}^{\infty} G(n, k) &= \sum_{n=0}^{\infty} \lim_{j \rightarrow \infty} G(n, k+j) \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \binom{2n}{n} \frac{1}{2^{2n}} = \frac{\arcsin(1)}{\pi} = \frac{1}{2}. \end{aligned}$$

Rearranging the resulting formula, and letting $k \rightarrow x$ completes the proof of (14).

The proof of (15) is basically identical to the above proof. Let us consider the following WZ-pair:

$$\begin{aligned} F(n, k) &= -U(n, k) \cdot \frac{4n}{2n+k}, \\ G(n, k) &= U(n, k) \cdot \frac{2(15n+2)(2n+1)^2 + kP(n, k)}{(2n+k+1)^2(2n+k)(2n+1)} \cdot \frac{k}{2}, \end{aligned} \quad (17)$$

where

$$U(n, k) = \frac{1}{16^n} \frac{\left(\frac{1}{2} + k\right)_n^2 \left(\frac{1}{2}\right)_n}{\left(1 + \frac{k}{2}\right)_n \left(\frac{1}{2} + \frac{k}{2}\right)_n (1)_n} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2},$$

and

$$P(n, k) = (2n + 1)(86n + 19) + 4k(20n + 7) + 12k^2.$$

Then $F(0, k) = 0$, $\lim_{n \rightarrow \infty} F(n, k) = 0$, and $\sum_{n=0}^{\infty} G(n, k)$ converges uniformly, so we have

$$\begin{aligned} \sum_{n=0}^{\infty} G(n, k) &= \sum_{n=0}^{\infty} \lim_{j \rightarrow \infty} G(n, k + j) \\ &= \frac{6}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n + 1)} \binom{2n}{n} \frac{1}{2^{4n}} = \frac{12 \arcsin(1/2)}{\pi} = 2. \end{aligned}$$

Rearranging the final result, and relabelling k as x completes the proof of (15). \square

Proof of Theorem 1. The shortest proof of (9) follows from using the definition of s , to show that $s = 1$ when $r = 1$. An alternative proof follows from using (14) to show that:

$$\begin{aligned} 2 + 2 \sum_{n=1}^{\infty} \frac{(4n + 1)}{(2n)(2n + 1)} \binom{2n}{n}^2 \frac{1}{2^{4n}} &= \lim_{x \rightarrow 0} \left(\pi \frac{\Gamma(x)\Gamma(x + 1)}{\Gamma^2(x + 1/2)} - \frac{1}{x} \right) \\ &= 4 \log(2). \end{aligned}$$

Similarly, (11) follows from using (15) to show that

$$\begin{aligned} 11 + 2 \sum_{n=1}^{\infty} \frac{(15n + 2)}{(2n)(2n + 1)} \binom{2n}{n}^2 \frac{1}{2^{8n}} &= \lim_{x \rightarrow 0} \left(4\pi \frac{\Gamma(x)\Gamma(x + 1)}{\Gamma^2(x + 1/2)} - \frac{4}{x} \right) \\ &= 16 \log(2). \end{aligned}$$

\square

Since the formulas in Theorem 2 involve Gamma functions, we can also use those identities to prove formulas for the Riemann zeta function. If we consider the Laurent expansion

$$\pi \frac{\Gamma(x)\Gamma(x + 1)}{\Gamma^2\left(x + \frac{1}{2}\right)} = \frac{1}{x} + 4 \log(2) + (-2\zeta(2) + 8 \log^2(2))x + \dots,$$

then by (14) we have

$$-\zeta(2) + 4 \log^2(2) = 2 \sum_{n=1}^{\infty} \frac{(4n + 1)}{(2n)(2n + 1)} \binom{2n}{n}^2 \frac{1}{2^{4n}} \left(-\frac{(2n + 1)}{(2n)(4n + 1)} + A_{2n} \right),$$

where A_n is the alternating harmonic series. We can also use a method from [13] to find formulas for $\zeta(3)$. Notice that Gosper first proved (20) (see [23] or [11]), and Batir proved (18) using log-sine integrals (combine formulas 3 and 4 on page 664 of [1]). Despite the fact that formula (19) is numerically true, it remains unproven.

Theorem 3. *The following identities are true:*

$$\zeta(3) = \frac{2}{7} \sum_{n=0}^{\infty} \frac{(4n+3)16^n}{(2n+1)^3(n+1)\binom{2n}{n}^2} \quad (18)$$

$$\zeta(3) \stackrel{?}{=} \frac{4}{7} \sum_{n=0}^{\infty} \frac{(3n+2)4^n}{(2n+1)^3(n+1)\binom{2n}{n}^2} \quad (19)$$

$$\zeta(3) = \frac{1}{16} \sum_{n=0}^{\infty} \frac{(30n+19)}{(2n+1)^3(n+1)\binom{2n}{n}^2} \quad (20)$$

Proof. The idea behind this theorem, is that shifting the summands in (9), (10), and (11) by $n \rightarrow n - 1/2$, changes those results into formulas for $\zeta(3)$. For instance, the summand in (10) becomes

$$\frac{(6n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{6n}} \rightarrow \frac{2(3n+2)4^n}{\pi^2(2n+1)^3(n+1)\binom{2n}{n}^2}.$$

In order to make this observation rigorous, we will prove (20) and (18) by using the WZ-pairs from Theorem 2. Additionally, a rigorous proof of (19) should be easy to construct by first finding a WZ proof of (10).

Let us shift both sides of (4) by $\frac{1}{2}$ and y , and then sum the equation from $n = 0$ to $n = \infty$. Under the hypothesis that $\lim_{n \rightarrow \infty} F(n + \frac{1}{2}, k + y) = 0$, we have

$$F\left(\frac{1}{2}, k + y\right) = - \sum_{n=0}^{\infty} G\left(n + \frac{1}{2}, k + y + 1\right) + \sum_{n=0}^{\infty} G\left(n + \frac{1}{2}, k + y\right).$$

The right-hand side of the formula telescopes with respect to k . Therefore, sum both sides of the equation from $k = 0$ to $k = \infty$:

$$\sum_{k=0}^{\infty} F\left(\frac{1}{2}, k + y\right) = - \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G\left(n + \frac{1}{2}, k + y\right) + \sum_{n=0}^{\infty} G\left(n + \frac{1}{2}, y\right).$$

In the cases we will consider, all three sums converge uniformly, the limit terms do not depend on y , and $G(n + \frac{1}{2}, 0) = 0$. Differentiating with respect to y , and using the notation $F^*(n, k) = \frac{\partial}{\partial k} F(n, k)$, and $G^*(n, k) = \frac{\partial}{\partial k} G(n, k)$, the formula becomes

$$\sum_{k=0}^{\infty} F^*\left(\frac{1}{2}, k\right) = \sum_{n=0}^{\infty} G^*\left(n + \frac{1}{2}, 0\right). \quad (21)$$

Notice that $G^*(n + \frac{1}{2}, 0) = \lim_{y \rightarrow 0} \frac{G(n + \frac{1}{2}, y)}{y}$.

If we use F and G from (16), then the various convergence requirements are satisfied, and we can show that

$$\sum_{n=0}^{\infty} G^*\left(n + \frac{1}{2}, 0\right) = \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{(4n+3)16^n}{(2n+1)^3(n+1)\binom{2n}{n}^2}.$$

The sum involving F^* is also easy to evaluate. First notice that

$$F\left(\frac{1}{2}, k\right) = -\frac{2}{\pi^2(2k+1)^2},$$

and therefore we can show that

$$\sum_{k=0}^{\infty} F^*\left(\frac{1}{2}, k\right) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \frac{7\zeta(3)}{\pi^2}.$$

Formula (18) follows from substituting these results into (21).

In order to prove (20), we will use the WZ-pair given in (17). Once again, all of the convergence requirements are satisfied. Therefore, observe that

$$\sum_{n=0}^{\infty} G^*\left(n + \frac{1}{2}, 0\right) = \frac{1}{4\pi^2} \sum_{n=0}^{\infty} \frac{(30n+19)}{(2n+1)^3(n+1)\binom{2n}{n}^2},$$

which matches the right-hand side of (20) up to a constant. In order to evaluate the F^* -sum, simply notice that

$$F\left(\frac{1}{2}, k\right) = \frac{-2}{\pi^2(k+1)^2},$$

and therefore

$$\sum_{k=0}^{\infty} F^*\left(\frac{1}{2}, k\right) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(k+1)^3} = \frac{4\zeta(3)}{\pi^2}.$$

Substituting these last two results into (21) completes the proof of (20). \square

We will conclude this section, by showing that it is also possible to find WZ-pairs when (8) diverges. If we consider the WZ-pair:

$$F(n, k) = \frac{n}{(2n+k)^2} \cdot \frac{\left(\frac{1}{2}\right)_n \left(1 + \frac{k}{2}\right)_n \left(\frac{1}{2} + \frac{k}{2}\right)_n}{(1)_n (1+k)_n^2} \cdot 16^n,$$

$$G(n, k) = -\frac{P(n, k)}{n(2n+k)^2(1+2n+k)} \cdot \frac{\left(\frac{1}{2}\right)_n \left(1 + \frac{k}{2}\right)_n \left(\frac{1}{2} + \frac{k}{2}\right)_n}{(1)_n (1+k)_n^2} \cdot 16^n,$$

where $P(n, k) = 3k^3 + k^2(20n+3) + kn(43n+12) + n^2(30n+11)$, then it is possible to obtain a finite summation identity

$$\begin{aligned} \sum_{n=1}^{m-1} \frac{30n+11}{(2n)(2n+1)} \binom{2n}{n}^2 &= -4 + 6 \sum_{n=1}^{m-1} \frac{1}{n} \binom{2n}{n} \\ &\quad + \frac{1}{2m} \binom{2m}{m}^2 {}_4F_3\left(\begin{smallmatrix} 1, 1, 2m, 2m \\ m+1, m+1, 2m+1 \end{smallmatrix}; 1\right), \end{aligned} \tag{22}$$

which holds for $m \in \mathbb{N}$. While this formula clearly corresponds to the values $(r, s) = (4, 1/11)$, it does not convey any new information about Mahler measures. Equation (13) already shows that if $r = 4$ then $s = 1/11$.

3. CONNECTIONS WITH THE ELLIPTIC DILOGARITHM

In the first section of the paper, we proved several formulas for $\log(2)$, which were equivalent to relations between Mahler measures. It is probably fortunate that the Mahler measure formulas were discovered first. It seems unlikely that equations (10) and (11) would have attracted much attention without Boyd's work. In order to provide some additional motivation, we will show that our method provides a new way to prove relations between elliptic dilogarithms. Let us briefly recall the definitions of $m(\alpha)$ and $n(\alpha)$:

$$\begin{aligned} m(\alpha) &:= m(\alpha + x + x^{-1} + y + y^{-1}), \\ n(\alpha) &:= m(x^3 + y^3 + 1 - \alpha xy). \end{aligned}$$

We examined $m(\alpha)$ in the previous section of this paper, and $n(\alpha)$ has been studied in papers such as [19], [20] and [16]. In the next theorem we will prove new q -series expansions for both functions.

Theorem 4. *Suppose that $q \in (-1, 1)$, and let $D(z) = \Im(\text{Li}_2(z) + \log|z| \log(1-z))$ denote the Bloch-Wigner dilogarithm. The following formulas are true:*

$$\frac{4}{\pi} \sum_{n=-\infty}^{\infty} D(iq^n) = m\left(4 \frac{\varphi^2(q)}{\varphi^2(-q)}\right), \quad (23)$$

$$\frac{9}{2\pi} \sum_{n=-\infty}^{\infty} D(e^{2\pi i/3} q^n) = n\left(3\sqrt[3]{x(q)}\right), \quad (24)$$

$$\frac{9}{\pi} \sum_{n=-\infty}^{\infty} D(e^{\pi i/3} q^n) = 2n\left(3\sqrt[3]{x(q)}\right) + n\left(3\sqrt[3]{x(q^2)}\right), \quad (25)$$

where $\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$, and $x(q) = 1 + 27q \prod_{n=1}^{\infty} \left(\frac{1 - q^{3n}}{1 - q^n}\right)^{12}$.

Proof. First notice that (24) implies (25). By the elementary functional equations for the Bloch-Wigner dilogarithm, $\frac{1}{2}D(z^2) = D(z) + D(-z)$, and $D(z) = -D(\frac{1}{z})$, it is possible to obtain

$$D(e^{\pi i/3} q^n) = D(e^{2\pi i/3} q^{-n}) + \frac{1}{2}D(e^{2\pi i/3} q^{2n}).$$

Summing over n shows that (25) follows easily from (24).

To prove (23) and (24), we will use an idea described in section 8 of [20]. Consider the following formula from [21] (Rodriguez-Villegas first proved a version of this formula in [19]):

$$\frac{\pi^2}{32x} m\left(4\sqrt{1 - \frac{\varphi^4(-q)}{\varphi^4(q)}}\right) = \sum_{\substack{n=0 \\ k=0}}^{\infty} \frac{(-1)^n (2n+1)}{((2n+1)^2 + x^2(2k+1)^2)^2},$$

where $q = e^{-\pi x}$, and $x > 0$. Express the sum as an integral, and then apply the involution for the weight-1/2 theta function:

$$\begin{aligned}
&= \frac{\pi^2}{16} \int_0^\infty u \left(\sum_{k=0}^\infty e^{-\pi(k+1/2)^2 x^2 u} \right) \left(\sum_{n=0}^\infty (-1)^n (2n+1) e^{-\pi(n+1/2)^2 u} \right) du \\
&= \frac{\pi^2}{32x} \int_0^\infty \sqrt{u} \left(\sum_{k=-\infty}^\infty (-1)^k e^{-\frac{\pi k^2}{x^2 u}} \right) \left(\sum_{n=0}^\infty (-1)^n (2n+1) e^{-\pi(n+1/2)^2 u} \right) du \\
&= \frac{\pi}{8x} \sum_{k=-\infty}^\infty (-1)^k \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)} \left(\frac{\pi|k|}{x} + \frac{1}{(2n+1)} \right) e^{-\pi(2n+1)|k|/x} \\
&= \frac{\pi}{8x} \sum_{k=-\infty}^\infty (-1)^k D(i e^{-\pi|k|/x}).
\end{aligned}$$

If we let $x \rightarrow 1/x$, and then use the following identity

$$\frac{\varphi^4(-q)}{\varphi^4(q)} = 1 - \frac{\varphi^4(-e^{-\pi/x})}{\varphi^4(e^{-\pi/x})},$$

it is easy to see that

$$m \left(4 \frac{\varphi^2(-q)}{\varphi^2(q)} \right) = \frac{4}{\pi} \sum_{k=-\infty}^\infty (-1)^k D(i q^{|k|}) = \frac{4}{\pi} \sum_{k=-\infty}^\infty D(i(-q)^k).$$

The second equality follows from the elementary identity $(-1)^k D(i q^{|k|}) = D(i(-q)^k)$. Formula (23) follows from sending $q \rightarrow -q$. Despite the fact that this last substitution is not rigorous, since we previously assumed that $x > 0$ and hence $q > 0$, the final result is true. A different proof follows from differentiating (23) with respect to q , and then applying the formulas of Ramanujan.

A proof of (24) can be obtained by looking at the following sum:

$$\sum_{n,k=-\infty}^\infty \frac{(3k+1)}{((3k+1)^2 + x^2(2n+1)^2)^2}.$$

Briefly, if this series is transformed into an integral of theta functions, then the involution for the weight-1/2 theta function leads to a dilogarithm sum, and the involution for the weight-3/2 theta function (essentially) leads to Rodriguez-Villegas's q -series for $n(\alpha)$ (see formula (2-10) in [16]). \square

For certain values of q , the left-hand sides of (23), (24), and (25) equal elliptic dilogarithms. In order to explain this statement, let us consider an elliptic curve

$$E : y^2 = 4x^3 - g_2x - g_3.$$

It is well known that E can be parameterized by $(x, y) = (\wp(u), \wp'(u))$, where $\wp(u)$ is the Weierstrass function. The periods of $\wp(u)$ are denoted by ω and ω' , and the

period ratio $\tau = \frac{\omega'}{\omega}$ is assumed to have $\Im(\tau) > 0$. If $P = (\wp(u), \wp'(u))$ denotes an arbitrary point on E , and $q = e^{2\pi i\tau}$, then the elliptic dilogarithm is defined by

$$D^E(P) := \sum_{n=-\infty}^{\infty} D(e^{2\pi i u/\omega} q^n).$$

Since we will only be interested in torsion points, we can assume that $u = a\omega + b\omega'$, for some $(a, b) \in \mathbb{Q}^2$. For appropriate choices of E , the series expansions in Theorem 4 equal $D^E(P)$ at 3, 4 and 6-torsion points.

Since our objective is to equate Mahler measures to elliptic dilogarithms, we need a method to identify the relevant elliptic curves. Let us define β using $\sqrt{1-\beta} = \varphi^2(-q)/\varphi^2(q)$. The classical theory of elliptic functions shows that we can calculate q as a function of β :

$$q = e^{2\pi i\tau} = \exp \left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \beta\right)} \right). \quad (26)$$

It is known that g_2 and g_3 are also functions of q [25]. In Ramanujan's notation we have $g_2 = \frac{4}{3}\pi^4 M(q)$, and $g_3 = \frac{8}{27}\pi^6 N(q)$ [2, pg. 126]. Applying formulas (13.3) and (13.4) in [2, pg. 127], we obtain:

$$J(\tau) = \frac{g_2^3}{g_3^3 - 27g_2^2} = \frac{(1 + 14\beta + \beta^2)^3}{108\beta(1-\beta)^4}. \quad (27)$$

In general, this relation will allow us to calculate β using g_2 and g_3 , and to recover g_2 and g_3 from values of β . Since there are six choices of β for each g_2^3/g_3^2 , caution must be exercised to pick the correct β . In many cases we checked our work by numerically computing q from g_2 and g_3 (with the Mathematica function “WeierstrassHalfPeriods”), and then comparing it to calculations using (26).

Theorem 5. *Let $E(k, \ell)$ denote the following elliptic curve:*

$$y^2 = 4x^3 - 27(k^4 - 16k^2 + 16)\ell^2 x + 27(k^6 - 24k^4 + 120k^2 + 64)\ell^3.$$

Formula (13) is equivalent to

$$11D^{E_1}(P_1) = 6D^{E_2}(P_2), \quad (28)$$

where $E_1 = E(5, 2)$, $E_2 = E(16, 1/2)$, $P_1 = (87, 1080)$, and $P_2 = (195, 432)$.

Formula (12) is equivalent to

$$5D^{E_3}(P_3) = 8D^{E_4}(P_4), \quad (29)$$

where $E_3 = E(8, 1/2)$, $E_4 = E(3\sqrt{2}, 1)$, $P_3 = (51, 216)$, and $P_4 = (33, 324)$.

Proof. If we set $\beta = 1 - 16/k^2$, then we can rearrange (27) to obtain

$$\frac{g_2^3}{g_3^2} = \frac{27(16 - 16k^2 + k^4)^3}{(64 + 120k^2 - 24k^4 + k^6)^2}.$$

Therefore, for some choice of ℓ , we have

$$\begin{aligned} g_2 &= 27(k^4 - 16k^2 + 16)\ell^2, \\ g_3 &= -27(k^6 - 24k^4 + 120k^2 + 64)\ell^3. \end{aligned}$$

In practice, ℓ will be chosen so that $E(k, \ell)$ has a fourth-degree torsion point P . We can then use $4\varphi^2(q)/\varphi^2(-q) = k$, along with equation (23), to conclude that

$$m(k) = D^{E(k, \ell)}(P).$$

We will begin by proving (28). It is easy to check that $E(5, 2)$ has a fourth-degree torsion point $P_1 = (87, 1080) = (\wp(\frac{\omega}{4}), \wp'(\frac{\omega}{4}))$. It follows from the definition of the elliptic dilogarithm, that

$$m(5) = \frac{4}{\pi} D^{E(5, 2)}(P_1).$$

A result from [16] shows that $m(1) + m(16) = 2m(5)$, and therefore we have proved that

$$m(1) + m(16) = \frac{8}{\pi} D^{E(5, 2)}(P_1). \quad (30)$$

When $k = 16$, it is easy to see that $E(16, 1/2)$ has a fourth-degree torsion point $P_2 = (195, 432) = (\wp(\frac{\omega}{4}), \wp'(\frac{\omega}{4}))$. Using the definition of the elliptic dilogarithm, we conclude that

$$m(16) = \frac{4}{\pi} D^{E(16, 1/2)}(P_2). \quad (31)$$

Substituting (30) and (31) into (13) completes the proof of (28).

Next we will prove (29). When $k = 8$, it is easy to check that $E(8, 1/2)$ has a fourth-degree torsion point $P_3 = (51, 216) = (\wp(\frac{\omega}{4}), \wp'(\frac{\omega}{4}))$. Thus, it follows that

$$m(8) = \frac{4}{\pi} D^{E(8, 1/2)}(P_3). \quad (32)$$

Finally, the elliptic curve $E(3\sqrt{2}, 1)$ has a fourth-degree torsion point $P_4 = (33, 324) = (\wp(\frac{\omega}{4}), \wp'(\frac{\omega}{4}))$. We can immediately conclude that $m(3\sqrt{2}) = \frac{4}{\pi} D^{E(3\sqrt{2}, 1)}(P_4)$. A result from [16] shows that $m(2) + m(8) = 2m(3\sqrt{2})$, and therefore we have proved that

$$m(2) + m(8) = \frac{8}{\pi} D^{E(3\sqrt{2}, 1)}(P_4). \quad (33)$$

Substituting (33) and (32) into (12) completes the proof of (29). \square

After considering Theorem 5, it seems likely that more formulas for elliptic dilogarithms will eventually be proved with the WZ method. The most interesting formulas are probably “exotic relations”, which have the form

$$\sum_r a_r D^E(rP) \stackrel{?}{=} 0.$$

Bloch and Grayson conjectured several exotic relations [7], and Zagier proposed a set of restrictions that E should satisfy in order to possess such a relation [10]. Theorem 4 shows that certain exotic relations are equivalent to formulas for hypergeometric functions. We will conclude this section by reformulating an exotic relation that Bertin proved in [5].

Theorem 6. *Let E denote the elliptic curve*

$$y^2 = 4x^3 - 432x + 1188,$$

and let $P = (-6, 54)$. Bertin's exotic relation

$$16D^E(P) - 11D^E(2P) = 0, \quad (34)$$

is equivalent to

$$16n \left(\frac{7 + \sqrt{5}}{\sqrt[3]{4}} \right) - 8n \left(\frac{7 - \sqrt{5}}{\sqrt[3]{4}} \right) = 19n \left(\sqrt[3]{32} \right). \quad (35)$$

Proof. If we notice that $P = (-6, 54) = (\wp(\frac{\omega-3\omega'}{6}), \wp'(\frac{\omega-3\omega'}{6}))$, then (34) is equivalent to

$$16 \sum_{n=-\infty}^{\infty} D(e^{\pi i/3} q^{n-1/2}) = 11 \sum_{n=-\infty}^{\infty} D(e^{2\pi i/3} q^n).$$

By formulas (24) and (25), this amounts to showing that

$$0 = 16n \left(3\sqrt[3]{x(\sqrt{q})} \right) - 19n \left(3\sqrt[3]{x(q)} \right) - 8n \left(3\sqrt[3]{x(q^2)} \right).$$

In order to calculate q , we will find it convenient to work in Ramanujan's theory of signature 3. If we assume that there exists a β such that

$$q = \exp \left(-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1 - \beta\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \beta\right)} \right),$$

then by formulas (4.6) and (4.8) in [4, pg. 107], and the values $g_2 = \frac{4}{3}\pi^4 M(q) = 432$, and $g_3 = \frac{8}{27}\pi^6 N(q) = -1188$, we have

$$\frac{6912}{6971} = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{(1 + 8\beta)^3}{64\beta(1 - \beta)^3},$$

and therefore $\beta = \frac{5}{32}$. We can also calculate $x(q)$ explicitly [4, pg. 104], and we find that

$$x(q) = \frac{1}{1 - \beta} = \frac{32}{27}.$$

Finally, if we write $x(\sqrt{q}) = \frac{1}{1 - \alpha}$ and $x(q^2) = \frac{1}{1 - \gamma}$, then α and γ are conjugate zeros of a second-degree modular polynomial with respect to β [16, pg. 94]:

$$27\alpha\beta(1 - \alpha)(1 - \beta) - (\alpha + \beta - 2\alpha\beta)^3 = 0.$$

Therefore, with the aid of a computer, we have $x(\sqrt{q}) = \frac{(7+\sqrt{5})^3}{108}$, and $x(q^2) = \frac{(7-\sqrt{5})^3}{108}$. □

Notice that (35) can be rewritten as an explicit series identity. By a result of Rodriguez-Villegas (see formula (2-36) in [16]), we have

$$3 \log \left(\frac{(7 + \sqrt{5})^{24}}{2^{53} 11^8} \right) = \sum_{n=1}^{\infty} \frac{(3n)!}{n \cdot n!^3} \left(16 \frac{2^{2n}}{(7 + \sqrt{5})^{3n}} - 8 \frac{2^{2n}}{(7 - \sqrt{5})^{3n}} - 19 \left(\frac{27}{32} \right)^n \right). \quad (36)$$

It would be extremely interesting to construct a WZ proof of (36). It seems likely that such a proof would require a more sophisticated argument than the proof of (11), since (36) involves irrational hypergeometric functions.

4. CONCLUSION

One of the main ideas underlying this work, is that elementary formulas often arise for very deep reasons. For example, Ramanujan's $1/\pi$ formulas follow from the theory of elliptic functions in a natural way. In the last few years, many of those formulas were reproved and *generalized* with WZ techniques. We have shown that formula (2) follows a similar pattern, since it follows from (15). In general, it is quite easy to discover formulas like (10) and (11) with the WZ method. Unfortunately, it is often very challenging to figure out why some of those results exist. The following infinite series is an example of a formula that we could not fully explain:

$$\frac{12}{\pi} \tan^{-1} \left(\frac{1}{\sqrt{2}} \right) = 3 - \sum_{n=1}^{\infty} \frac{54n^2 + n - 1}{(3n-1)(3n+1)(4n-1)} \binom{2n}{n} \binom{4n}{2n} \frac{1}{2^{6n}}. \quad (37)$$

Based on the derivations of (10) and (11), it is possible that (37) is somehow related to higher L -values. A key challenge is therefore to determine if any L -functions are associated to these sorts of higher hypergeometric functions.

A second important point is that formula (8) involves two q -parameters. In some sense, equations (9) through (11) can be regarded as more complicated analogues of Ramanujan's formulas for $\frac{1}{\pi}$. One of Ramanujan's major insights was to find formulas such as

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (An + B) \binom{2n}{n}^3 X^n,$$

where A , B , and X are parameterized by theta functions. Since the various theta functions are algebraically dependent, the problem of producing rational formulas for $1/\pi$ amounts to finding rational points on algebraic curves. An easy corollary is to also obtain formulas where X is algebraic and irrational. In order to illustrate the more difficult nature of formulas such as (10) and (11), we will briefly consider equation (23). Using that formula, it is possible to find q -parameterizations for the r and s values in equation (8):

$$r = \frac{\varphi^2(-q)}{\varphi^2(q)}, \quad s = \frac{\mathcal{L}(i, q)}{\mathcal{L}(i, -q)}, \quad (38)$$

where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \mathcal{L}(i, q) = \sum_{n=-\infty}^{\infty} D(iq^n).$$

Based on these parametric formulas, we conjecture that r and s are algebraically independent for almost all values of $q \in (0, 1)$. Furthermore, it seems plausible that other WZ identities, such as the $1/\pi^2$ formulas [12], might arise in a similar manner. The largest hurdle in testing such a hypothesis, is to identify functions like the elliptic dilogarithm, by starting from sporadically occurring formulas such as (10) and (11).

REFERENCES

- [1] N. Batir, Integral representations of some series involving $\binom{2k}{k}^{-1} k^{-n}$ and some related series, *Applied Math. and Comp.* **147** (2004), 645-667.
- [2] B. C. Berndt, Ramanujan's Notebooks, Part III, *Springer-Verlag, New York*, 1991.
- [3] B. C. Berndt, Ramanujan's Notebooks, Part IV, *Springer-Verlag, New York*, 1994.
- [4] B. C. Berndt, Ramanujan's Notebooks, Part V, *Springer-Verlag, New York*, 1998.
- [5] M. J. Bertin, Mesure de Mahler et régulateur elliptique: preuve de deux relations "exotiques". *Number theory*, 1-12, CRM Proc. Lecture Notes, 36, *Amer. Math. Soc., Providence, RI*, 2004.
- [6] M. J. Bertin, Mesure de Mahler d'une famille de polynômes. *J. Reine Angew. Math.* 569 (2004), 175-188.
- [7] S. Bloch and D. Grayson, K_2 and L -functions of elliptic curves: computer calculations. Applications of algebraic K -theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), 79-88, *Contemp. Math.*, 55, *Amer. Math. Soc., Providence, RI*, 1986.
- [8] D. W. Boyd, Mahler's measure and special values of L -functions, *Experiment. Math.* **7** (1998), 37-82. *Academic Press*, 1994.
- [9] C. Deninger, Deligne periods of mixed motives, K -theory and the entropy of certain Z^n -actions, *J. Amer. Math. Soc.* **10** (1997), no. 2, 259-281.
- [10] A. B. Goncharov and A. M. Levin, Zagier's conjecture on $L(E, 2)$. *Invent. Math.* 132 (1998), no. 2, 393-432.
- [11] R. W. Gosper, "Strip Mining in the Abandoned Orefields of Nineteenth Century Mathematics." In *Computers in Mathematics* (Ed. D. V. Chudnovsky and R. D. Jenks). *New York: Dekker*, 1990.
- [12] J. Guillera, About a new kind of Ramanujan-type series. *Experiment. Math.* 12 (2003), no. 4, 507-510.
- [13] J. Guillera, Hypergeometric identities for 10 extended Ramanujan-type series. *Ramanujan J.* **15** (2008), no. 2, 219-234.
- [14] N. Kurokawa and H. Ochiai, Mahler measures via crystalization, *Commentarii Mathematici Universitatis Sancti Pauli*, **54** (2005), 121-137.
- [15] M. N. Lalin, On a conjecture of Boyd, to appear in the *International Journal of Number Theory*.
- [16] M. N. Lalin and M. D. Rogers, Functional equations for Mahler measures of genus-one curves, *Algebra and Number Theory*, **1** (2007), no. 1, 87-117.
- [17] M. Petkovsek, H.S. Wilf and D. Zeilberger: *A=B*. *Peters A.K.: Ltd.*, (1996).
- [18] F. Rodriguez-Villegas, Identities between Mahler measures, *Number theory for the millennium, III (Urbana, IL, 2000)*, 223-229, *A K Peters, Natick, MA*, 2002.
- [19] F. Rodriguez-Villegas, Modular Mahler measures I, *Topics in number theory* (University Park, PA, 1997), 17-48, *Math. Appl.*, 467, *Kluwer Acad. Publ., Dordrecht*, 1999.
- [20] M. D. Rogers, Hypergeometric formulas for lattice sums and Mahler measures, Submitted.
- [21] M. D. Rogers and B. Yttanan, Somos's modular equations and lattice sums, Submitted.

- [22] J. Voight, Aspects of complex multiplication, Course notes taken from a seminar taught by Don Zagier.
- [23] E. W. Weisstein, “Apery’s Constant.” From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/AperysConstant.html>
- [24] E. W. Weisstein, “Dougall’s Theorem.” From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/DougallsTheorem.html>
- [25] E. W. Weisstein, “Elliptic Invariants.” From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/EllipticInvariants.html>
- [26] H.S. Wilf and D. Zeilberger, Rational functions certify combinatorial identities, Journal Amer. Math. Soc. 3, 147-158, (1990). (Winner of the Steele prize).

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